Some notes on maximal number of cycles in reflexive cacti

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Introduction

- G is a connected simple graph
- Characteristic polynomial $P_G(\lambda) = \det(\lambda I - A)$, A is the adjacency matrix
- Its roots are all real numbers and we assume their non-increasing order

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n$$

- For connected graphs $\lambda_1 > \lambda_2$ holds

- A graph is treelike, or a cactus, if its cycles have no common edges
- Cycles of multicyclic cactus form a bundle if all of them contain the same vertex
- A cycle of a multicyclic cactus is free if only one of its vertices has the degree greater than 2

In this work we analyze a class of multicyclic cacti whose second largest eigenvalue λ_2 does not exceed 2.

Graphs whose second largest eigenvalue is bounded by 2 appear in the theory of reflection groups, and, therefore, they are called **reflexive.**

Some classes of multicyclic reflexive cacti have been described in previous work. They have been considered under some conditions, among which was the condition that their cycles do not form a bundle, and it has been shown that such graphs have at most 5 cycles.

Though one special class of cacti with the bundle has been described previously, for the first time multicyclic reflexive cacti whose cycles do form a bundle are considered here in general.

We will find the maximum number of cycles in these cacti whenever that number is finite.

Some auxiliary and former results

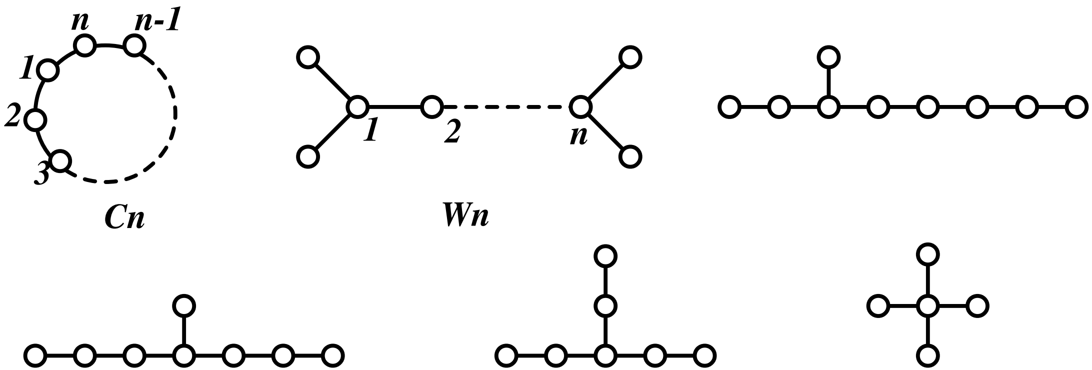
Lemma. (Schwenk) Given a graph *G*, let $C(v)$ $(C(uv))$ denote the set of all its cycles containing a vertex *v* (resp. an edge *uv*). Then

1. $$P_G(\lambda) = \lambda P_{G-v}(\lambda) - \sum_{u \in Adj(v)} P_{G-v-u}(\lambda) - 2 \sum_{C \in C(v)} P_{G-V(C)}(\lambda),$$

2. $$P_G(\lambda) = P_{G-uv}(\lambda) - P_{G-v-u}(\lambda) - 2 \sum_{C \in C(uv)} P_{G-V(C)}(\lambda),$$

where *Adj(v)* denotes the set of neighbours of *v,* while *G - V(C)* is the graph obtained from *G* by removing the vertices belonging to the cycle *C.*

By the Interlacing theorem, the property $\lambda_2(G) \leq 2$ is a hereditary one (if G has this property, then every subgraph H has it, too). Smith graphs are maximal connected graphs for the property $\lambda_1(G) \leq 2$. (For all of them $\lambda_1 = 2$ holds)



Smith graphs

RS-Theorem. Let *G* be a graph with a cut-vertex *v*.

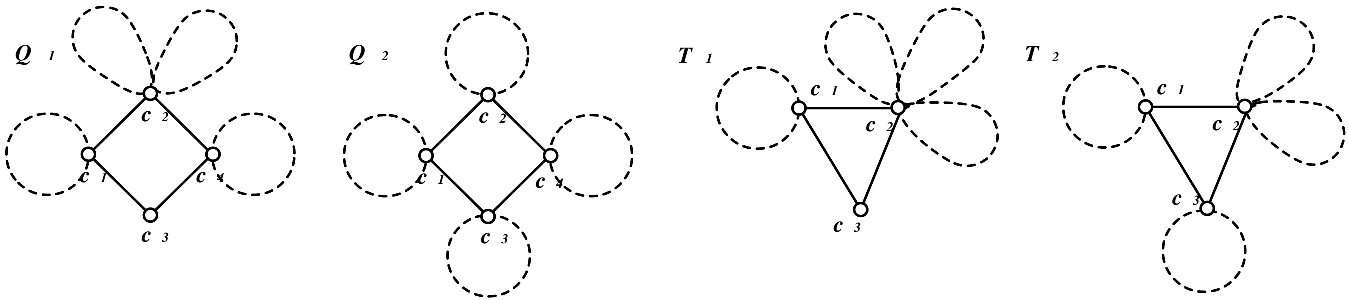
1) If at least two components of $G - v$ are supergraphs of Smith graphs, and if at least one of them is a proper supergraph, then $\lambda_2(G) > 2$ holds.

2) If at least two components of $G - v$ are Smith graphs, and the rest are subgraphs of Smith graphs, then $\lambda_2(G) = 2$ holds.

3) If at most one component of $G - v$ is Smith graph, and the rest are proper subgraphs of Smith graphs, then $\lambda_2(G) < 2$ holds.

This theorem cannot tell whether *G* is reflexive or not if, after removing the cut-vertex, one of the components is a supergraph of some Smith graph and all others are subgraphs of some Smith graphs. Graphs like these we call RS-undecidable; otherwise, they are RS-decidable.

Theorem. A treelike reflexive graph to which RS-theorem cannot be applied and whose cycles do not form a bundle has at most five cycles. The only such graphs with five cycles, which are all maximal, i.e. cannot be extended at any vertex, are the four families of graphs Q_1, Q_2, T_1 and T_2.



(a) (b) (c) (d)

A bundle of cycles – minimal components

In this work we determine the maximum number of cycles for maximal reflexive RS-undecidable cacti whose cycles form a bundle, and therefore we find the maximum number of cycles for all maximal reflexive RS-undecidable cacti.

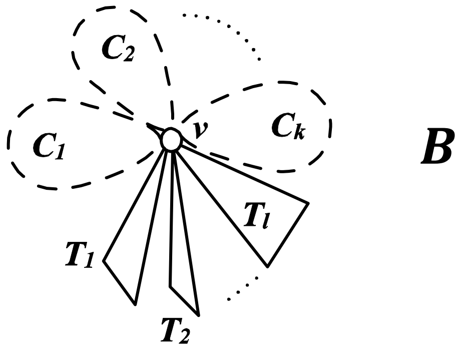


Figure 1.

Let B be a cactus (Figure 1) with k cycles that make a bundle and let the vertex v be the cut-vertex that belongs to all cycles.

The vertices of cycles that are adjacent to the vertex v we call black vertices and all vertices of cycles different from black vertices and the vertex v we call white vertices.

Every vertex of B may be additionally loaded by some tree.

If there are trees which are leaned to the vertex v, we denote them by $T_1,...,T_m$. Let $C_1,...,C_k$ be unicyclic subgraphs of B that contain cycles of length $n_1,...,n_k$, respectively, and in every such subgraph the degree of v is 2.

Therefore, the graph $B-v$ contains the components $K_i = C_i - v$ $(i=1,...,k)$ and the components $L_j = T_j - v$ $(j=1,...,m)$.

If all of the components $K_1,...K_k, L_1,...,L_m$ are Smith graphs, or subgraphs of some Smith graphs, then the graph B is RS-decidable and reflexive.

In such a graph, maximum number of cycles does not exist. The number of cycles can always be increased, because by adding a new cycle leaned on a vertex v, the fact that B is RS-decidable and reflexive does not change.

By G_∞ we denote the family of all such cacti. From now on we consider graphs that do not belong to the family G_∞.

If the graph B does not belong to the family G_∞, it can be reflexive only if it is RS-undecidable.

Then, one of the components of the graph $B - v$ must be a proper supergraph of a Smith tree, while all others are proper subgraphs of some Smith trees.

In order to find maximum number of cycles it is sufficient to consider, in a way, minimal cases of graphs. i.e. some characteristic graphs that are subgraphs of all graphs of type *B*.

G_1 -type

We say that graph G, which has the cyclic structure like graph B (Figure 1) is G_1-type graph if the following conditions are satisfied:

1.1. All cycles of the graph G are free.

1.2. There is only one L-component, for example L_1 ($L_1 = T_1 - v$), and it is a supergraph of some Smith tree.

1.3. For every vertex u of the component L_1, which has the degree 1 in the graph G, the condition $\lambda_1(L_1 - u) \leq 2$ holds.

G_2 -type

We say that graph G which has the cyclic structure like graph B (Figure 1) is G_2 -type graph if the following conditions are satisfied:

2.1. There is no tree leaned on the vertex v.

2.2. One of the K-components, for example K_1 ($K_1 = C_1 - v$) is a supergraph of some Smith tree, while all other K- components are paths.

2.3. For every vertex u of the component K_1, which has the degree 1 in the graph G, the condition $\lambda_1(K_1 - u) \leq 2$ holds.

Theorem 1. Let G be a reflexive graph with the cyclic structure as of graph B (Figure 1), that does not belong to the family G_∞. Then, G contains as a subgraph either a G_1-type graph or a G_2-type graph.

Theorem 2. 1) Let the graph G be the G_1-type graph. Then, it is reflexive if and only if the following condition holds $P_{T_1}(2) - 2kP_{L_1}(2) \leq 0$.

2) Let the graph G be the G_2-type graph. Then, it is reflexive if and only if the following condition holds: $P_{C_1}(2) - 2(k-1)P_{K_1}(2) \leq 0$.

Proof. In these two cases, the assessment of the maximum number of cycles in graph G (G_1-type or G_2-type) is based on the examination of the sign of $P_G(2)$. Let $H = G - u$, where u is the vertex of the component L_1 (K_1) of the graph $G - v$, whose degree is 1 in the graph G. By RS-theorem

$\lambda_2(H) < 2$ and $P_H(2) < 0$ hold, because $\lambda_1(L_1 - u) \le 2$ ($\lambda_1(K_1 - u) \le 2$) holds. Therefore, using the Interlacing theorem, we get $P_G(2) \le 0 \Leftrightarrow \lambda_2(G) \le 2$, so now we shall calculate $P_G(2)$ for both our cases.

Let G be G_1-type graph. Applying Scwenk's lemma to the vertex v we get

$$P_G(2) = 2n_1...n_k P_{L_1}(2) - 2P_{L_1}(2)((n_1-1)n_2...n_k + n_1(n_2-1)...n_k + ... + n_1...n_{k-1}(n_k-1)) -$$
$$P_{L_1-b}(2)n_1...n_k - 2P_{L_1}(2)(n_2...n_k + n_1n_3...n_k + ... + n_1...n_{k-1}) = n_1...n_k(2(1-k)P_{L_1}(2) - P_{L_1-b}(2)),$$

(b is the black vertex of the tree T_1) and by applying it to the vertex v and the graph T_1 we get $P_{T_1}(2) = 2P_{L_1}(2) - P_{L_1-b}(2)$.

Therefore, the condition $P_G(2) \le 0$ becomes equivalent to the condition $P_{T_1}(2) - 2kP_{L_1}(2) \le 0.$

Let G be G_2 -type graph. Applying Scwenk's lemma to the vertex v we get

$$\begin{aligned}
&P_G(2) = 2P_{K_1}(2)n_2 \ldots n_k - (P_{K_1-b_1}(2) + P_{K_1-b_2}(2))n_2 \ldots n_k - \\
&2P_{K_1}(2)((n_2-1)n_3 \ldots n_k + n_2(n_3-1) \ldots n_k + \ldots + n_2 \ldots n_{k-1}(n_k-1)) - \\
&2P_{C_1-C}(2)n_2 \ldots n_k - 2P_{K_1}(2)(n_2 \ldots n_k + n_1 n_3 \ldots n_k + \ldots + n_1 \ldots n_{k-1}) = \\
&n_2 \ldots n_k(2P_{K_1}(2) - P_{K_1-b_1}(2) - P_{K_1-b_2}(2) - 2(k-1)P_{K_1}(2) - 2P_{C_1-C}(2))
\end{aligned}$$

where b_1 and b_2 are black vertices of the unicyclic subgraph C_1 and C denotes the cycle of the length n_1 that belongs to the subgraph C_1; and applying Schwenk's lemma to the vertex v and the graph C_1, we get

$$P_{C_1}(2) = 2P_{K_1}(2) - P_{K_1-b_1}(2) - P_{K_1-b_2}(2) - 2P_{C_1-C}(2)$$

and, therefore, the condition $P_G(2) \leq 0$ becomes equivalent to the condition $P_{C_1}(2) - 2(k-1)P_{K_1}(2) \leq 0$.□

Now we shall discuss the structure of the component K_1 (L_1). The component which is a supergraph of some Smith tree must contain at least one of the trees $F_1, \ldots, F_9$ (Figure 2), because they are minimal forbidden trees for the property $\lambda_1 \leq 2$.

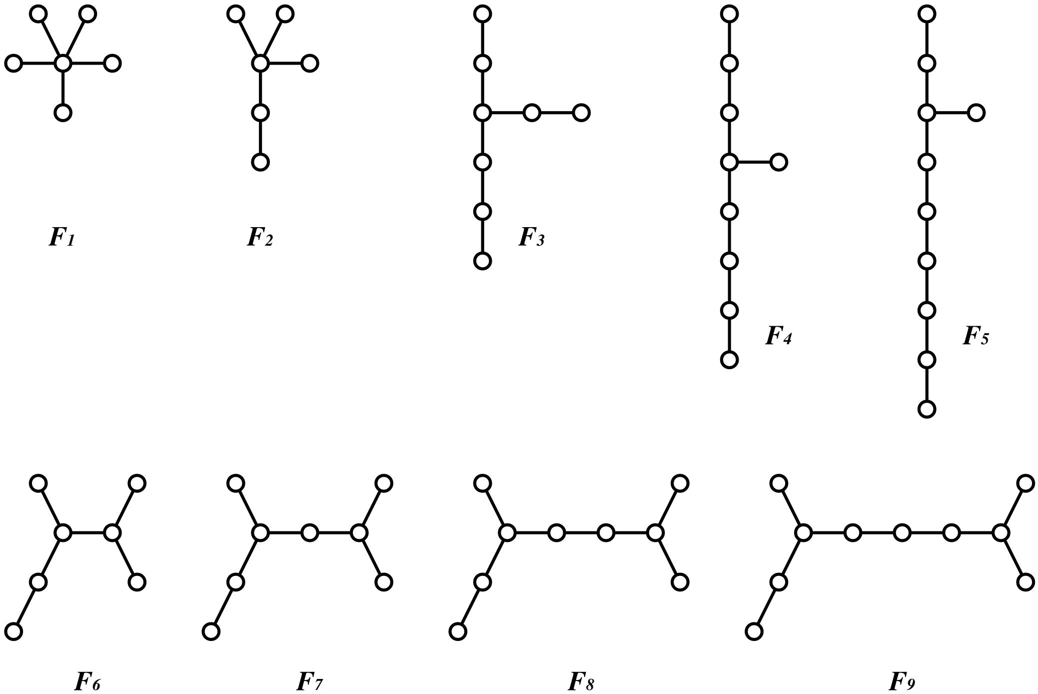


Figure 2: Minimal forbidden trees for the property $\lambda_1 \leq 2$

Vertices of the component K_1 (i.e. L_1) adjacent to the vertex v in graph G, which is G_1-type (G_2-type), are denoted by b_1 and b_2 (i.e. b) and they are called the black vertices. Let the component K_1 (i.e. L_1) contain a tree F ($F \in \{F_1, ..., F_9\}$). Then at least one of the vertices that belong to F must be the black vertex of the corresponding component, otherwise removing the black vertices from the graph G produces *RS*-decidable graph for which $\lambda_2(G - b_1 - b_2) > 2$ (i.e. $\lambda_2(G - b) > 2$) holds and therefore $\lambda_2(G) > 2$ holds.

A. G is G_1 - type

In this case, the black vertex b of the component L_1 must belong to the tree F_i ($i = 1,...,9$) contained in the component, so it is sufficient to analyze the cases when $L_1 = F_i$. Further, the vertex x cannot be the black vertex, because otherwise graph $G - b$ becomes RS-decidable and nonreflexive. However, any other vertex of the tree F_i different from x may be the black vertex.

B. G is G_2 -type graph

If graph F ($F \in \{F_1, ..., F_9\}$), subgraph of the component K_1, contains both black vertices of the component K_1, then $K_1 = F$.

If graph F ($F \in \{F_1, ..., F_9\}$), subgraph of the component K_1, contains only one black vertex (for example b_1) of this component, then K_1 can be presented as F, extended in such way that in K_1 exists the pendant edge, not belonging to F, whose end is the other black vertex b_2 . Now we discuss the place of the vertex x in K_1 and we see 4 possibilities:

A) *x is not one of the vertices of cycle and the degree of x is greater than 1.*

The component K_1 can be minimized by deleting the tree, disjoint from *F*, which is leaned on the vertex *x*, so in order to find maximal number of cycles in *G* we can ignore this case.

B) *x is not one of the vertices of cycle and the degree of x is 1.*

$K_1 - x$ is still the supergraph of some Smith tree and the condition $\lambda_1(K_1 - x) \leq 2$ does not hold, so we reject this case , too.

C) *x is* b_1.

$K_1 - b_1 - b_2$ is a supergraph of some Smith tree and then, by applying RS-theorem, $\lambda_2(G - b_1 - b_2) > 2$ holds, i.e. *G* is not reflexive; or $K_1 - b_1 - b_2$ is one of the Smith trees and then, by applying RS-theorem, we get $\lambda_2(G - b_1 - b_2) \geq 2$ wherein the equality holds (and *G* is reflexive) only in case that G contains only two cycles, and that is the minimal number of cycles in G, so this case is not of interest, too.

D) *x is one of the vertices of cycle different from black vertices (and, of course, different from v).*

The degree of x in K_1 is at least 2 and in this case we can say that $K_1 = F \cdot P_{i+1}$ (coalescence is at vertex x), where P_{i+1} is the path of the length i which belongs to the cycle and connects x with b_2 and its vertices are $x, x_1, \ldots, x_{i-1}, x_i = b_2$. Now, if we remove the vertex b_2 from the graph G, we get the graph $G - b_2$ which is a supergraph (proper or not) of some G_1-type graph. However, the number of cycles in the graph G cannot exceed the number of cycles in the corresponding G_1-type graph, so this case is not of interest for us.

Results

Based on Theorem 2 we can determine the maximum number k of cycles in a bundle by discussing possible cases.

First we show Table 1, for the cases $L_1 = F$ ($F \in \{F_1, \ldots, F_9\}$).

F	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
$k_{\max}$	2	4	7	11	22	4	4	4	4

Table 1

The detailed description is given only for the graph F_4 in Table 2, as an example. The black vertex b is one of the vertices of the graph F different from the vertex x and the values $P_{T_1}(2)$, $P_{T_1-v}(2)$ and $P_{T_1}(2) - 2(k-1)P_{T_1-v}(2)$ are also listed.

F_4	b	$P_{T_1}(2)$	$P_{T_1-v}(2)$	$P_{T_1}(2)-2kP_{T_1-v}(2)$	$k_{\max}$
	s_1	-8	-2	$4(k-2)$	**2**
	s_2	-16	-2	$4(k-4)$	**4**
	s_3	-28	-2	$4(k-7)$	**7**
	s_4	-44	-2	$4(k-11)$	**11**
	s_5	-25	-2	$4k-25$	**6***
	s_6	-12	-2	$4(k-3)$	**3**
	s_7	-5	-2	$4k-5$	**1***
	s_8	-13	-2	$4k-13$	**3***

Table 2

The numbers with the asterisk stand for the cases where the strict inequality $P_{T_1}(2) - 2kP_{L_1}(2) < 0$ holds and in the unmarked cases the equality is reached.

In the next table we show the results for the cases when $K_1 = F$ ($F \in \{F_1, ..., F_9\}$). The details of computations we show only for the case $K_1 = F_2$ in Table 4, as an example.

F	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
$k_{\max}$	4	10	20	32	74	13	13	13	13

Table 3

In Table 4 vertices that are identified with black vertices are listed in column (b_1, b_2), and in other columns we show the values $P_{C_1}(2)$, $P_{C_1-v}(2)$ and $P_{C_1}(2) - 2(k-1)P_{C_1-v}(2)$, as well as maximal value of *k*, based on the condition (1).

F_2	(b_1,b_2)	$P_{C_1}(2)$	$P_{C_1-v}(2)$	$P_{C_1}(2)-2(k-1)P_{C_1-v}(2)$	$k_{\max}$
	(x,s_1)	-24	-4	$8(k-4)$	**4**
	(x,s_2)	-48	-4	$8(k-7)$	**7**
	(x,s_3)	-20	-4	$4(2k-7)$	**3***
	(s_1,s_2)	-72	-4	$8(k-10)$	**10**
	(s_1,s_3)	-36	-4	$4(2k-11)$	**5***
	(s_2,s_3)	-60	-4	$4(2k-17)$	**8***
	(s_3,s_4)	-28	-4	$4(2k-9)$	**4***

Table 4

Theorem 4. The maximum number of cycles in a G_1-type graph is 22.

Theorem 5. The maximum number of cycles in a G_2-type graph is 74.

From the two previous theorems follows the next one.

Theorem 6. The maximum number of cycles in RS-undecidable reflexive graph with a bundle is 74.

Now we can state the main result.

Theorem 7. The maximum number of cycles in RS-undecidable reflexive graph is 74.

(As mentioned before, the number of cycles in RS-decidable cacti is not limited.)

THANK YOU!